Let f be a function satisfying $f(xy) = f(x)/y$ for all positive real numbers x and y. If $f(500) = 3$, what is the value of $f(600)$?

(A) 1 **(B)** 2 **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** $\frac{18}{5}$

2001 AMC 12, Problem #9—

"$600 = 500 \times \frac{6}{5}$."

Solution

(C) Note that

$$f(600) = f(500 \cdot \frac{6}{5}) = \frac{f(500)}{6/5} = \frac{3}{6/5} = \frac{5}{2}.$$

OR

For all positive x,

$$f(x) = f(1 \cdot x) = \frac{f(1)}{x},$$

so $xf(x)$ is the constant $f(1)$. Therefore,

$$600f(600) = 500f(500) = 500(3) = 1500,$$

so $f(600) = \frac{1500}{600} = \frac{5}{2}$. Note. $f(x) = \frac{1500}{x}$ is the unique function satisfying the given conditions.

Difficulty: Hard

NCTM Standard: Algebra Standard for Grades 9–12: Represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Calculus and Analysis > Functions

The parabola with equation $y = ax^2 + bx + c$ and vertex (h, k) is reflected about the line $y = k$. This results in the parabola with equation $y = dx^2 + ex + f$. Which of the following equals $a + b + c + d + e + f$?

(A) $2b$ **(B)** $2c$ **(C)** $2a + 2b$ **(D)** $2h$ **(E)** $2k$

2001 AMC 12, Problem #13—
"The reflection of a point (x, y) about the line $y = k$ is $(x, 2k - y)$."

Solution
(E) The equation of the first parabola can be written in the form

$$y = a(x - h)^2 + k = ax^2 - 2axh + ah^2 + k,$$

and the equation for the second (having the same shape and vertex, but opening in the opposite direction) can be written in the form

$$y = -a(x - h)^2 + k = -ax^2 + 2axh - ah^2 + k.$$

Hence,

$$a+b+c+d+e+f = a+(-2ah)+(ah^2+k)+(-a)+(2ah)+(-ah^2+k) = 2k.$$

OR

The reflection of a point (x, y) about the line $y = k$ is $(x, 2k - y)$. Thus, the equation of the reflected parabola is

$$2k - y = ax^2 + bx + c, \text{ or equivalently, } y = 2k - (ax^2 + bx + c).$$

Hence $a + b + c + d + e + f = 2k$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grades 9–12: Apply transformations and use symmetry to analyze mathematical situations.
Mathworld.com Classification: Geometry > Transformations > Reflections

Given the nine-sided regular polygon $A_1A_2A_3A_4A_5A_6A_7A_8A_9$, how many distinct equilateral triangles in the plane of the polygon have at least two vertices in the set $\{A_1, A_2, \ldots A_9\}$?

(A) 30 **(B)** 36 **(C)** 63 **(D)** 66 **(E)** 72

2001 AMC 12, Problem #14—

"Each of the $\binom{9}{2} \equiv 9C2 = 36$ pairs of vertices determines two equilateral triangles."

Solution

(D) Each of the $\binom{9}{2} \equiv 9C2 = 36$ pairs of vertices determines two equilateral triangles, for a total of 72 triangles. However, the three triangles $A_1A_4A_7$, $A_2A_5A_8$, and $A_3A_6A_9$ are each counted 3 times, resulting in an overcount of 6. Thus, there are 66 distinct equilateral triangles.

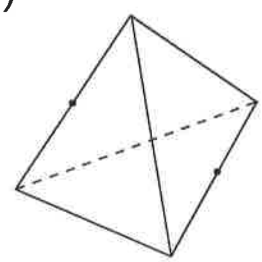
Difficulty: Hard

NCTM Standard: Algebra Standard for Grades 9–12: Understand patterns, relations, and functions.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Triangle

Discrete Mathematics > Combinatorics > Permutations > Arrangement

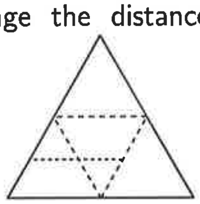
An insect lives on the surface of a regular tetrahedron with edges of length 1. It wishes to travel on the surface of the tetrahedron from the midpoint of one edge to the midpoint of the opposite edge. What is the length of the shortest such trip? (Note: Two edges of a tetrahedron are *opposite* if they have no common endpoint.)



(A) $\frac{1}{2}\sqrt{3}$ **(B)** 1 **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$ **(E)** 2

2001 AMC 12, Problem #15—
"Unfold the tetrahedron onto a plane."

Solution
(B) Unfold the tetrahedron onto a plane. The two opposite-edge midpoints become the midpoints of opposite sides of a rhombus with sides of length 1, so are now 1 unit apart. Folding back to a tetrahedron does not change the distance and it remains minimal.



Difficulty: Medium-hard
NCTM Standard: Geometry Standard for Grades 9–12: Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Solid Geometry > Polyhedra > Tetrahedra

A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

(A) $8!$ **(B)** $2^8 8!$ **(C)** $(8!)^2$ **(D)** $\frac{16!}{2^8}$ **(E)** $16!$

2001 AMC 12, Problem #16—

"There are $16!$ permutations of the sixteen symbols(both socks and shoes)."

Solution

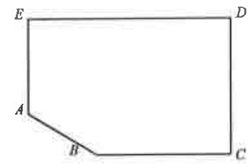
(D) Number the spider's legs from 1 through 8, and let a_k and b_k denote the sock and shoe that will go on leg k. A possible arrangement of the socks and shoes is a permutation of the sixteen symbols $a_1, b_1, \ldots a_8, b_8$, in which a_k precedes b_k for $1 \le k \le 8$. There are 16! permutations of the sixteen symbols, and a_1 precedes b_1 in exactly half of these, or $16!/2$ permutations. Similarly, a_2 precedes b_2 in exactly half of those, or $16!/2^2$ permutations. Continuing, we can conclude that a_k precedes b_k for $1 \le k \le 8$ in exactly $16!/2^8$ permutations.

Difficulty: Hard

NCTM Standard: Algebra Standard for Grades 9–12: Represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Discrete Mathematics > Combinatorics > Permutations

A point P is selected at random from the interior of the pentagon with vertices $A = (0,2)$, $B = (4,0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$, and $E = (0,4)$. What is the probability that $\angle APB$ is obtuse?



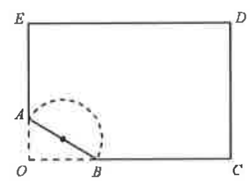
(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{5}{16}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{2}$

2001 AMC 12, Problem #17—

"Since $\angle APB = 90°$ if and only if P lies on the semicircle with center $(2,1)$ and radius $\sqrt{5}$, the angel is obtuse if and only if the point P lies inside this semicircle."

Solution

(C) Since $\angle APB = 90°$ if and only if P lies on the semicircle with center $(2,1)$ and radius $\sqrt{5}$, the angel is obtuse if and only if the point P lies inside this semicircle. The semicircle lies entirely inside the pentagon, since the distance, 3, from $(2,1)$ to $\overline{DE}$ is greater than the radius of the circle. Thus the probability that the nagle is obtuse is the ratio of the area of the semicircle to the area of the pentagon.



Let $O = (0,0)$, $A = (0,2)$, $B = (4,0)$, $C = (2\pi+1,0)$, $D = (2\pi+1,4)$, and $E = (0,4)$. Then the area of the pentagon is

$$[ABCDE] = [OCDE] - [OAB] = 4 \cdot (2\pi + 1) - \frac{1}{2}(2 \cdot 4) = 8\pi,$$

and the area of the semicircle is

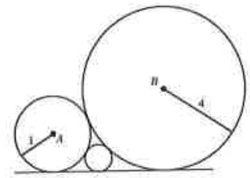
$$\frac{1}{2}\pi(\sqrt{5})^2 = \frac{5}{2}\pi.$$

The probability is

$$\frac{\frac{5}{2}\pi}{8\pi} = \frac{5}{16}.$$

Difficulty: Hard
NCTM Standard: Geometry Standard for Grades 9–12: Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Obtuse Triangle

A circle centered at A with a radius of 1 and a circle centered at B with a radius of 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as shown. The radius of the third circle is



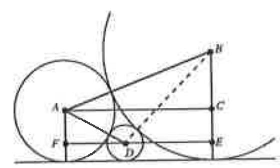
(A) $\frac{1}{3}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{5}{12}$ **(D)** $\frac{4}{9}$ **(E)** $\frac{1}{2}$

2001 AMC 12, Problem #18—

"Connects the center of the circles, and explore the relationships."

Solution

(D) Let C be the intersection of the horizontal line through A and the vertical line through B. In right triangle ABC, we have $BC = 3$ and $AB = 5$, so $AC = 4$. Let x be the radius of the third circle, and D be the center. Let E and F be the points of intersection of the horizontal line through D with the vertical lines through B and A, respectively, as shown.



In $\triangle BED$ we have $BD = 4 + x$ and $BE = 4 - x$, so

$$DE^2 = (4+x)^2 - (4-x)^2 = 16x,$$

and $DE = 4\sqrt{x}$. In $\triangle ADF$ we have $AD = 1 + x$ and $AF = 1 - x$, so

$$FD^2 = (1+x)^2 - (1-x)^2 = 4x,$$

and $FD = 2\sqrt{x}$. Hence,

$$4 = AC = FD + DE = 2\sqrt{x} + 4\sqrt{x} = 6\sqrt{x}$$

and $\sqrt{x} = \frac{2}{3}$, which implies $x = \frac{4}{9}$.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grades 9–12: Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Plane Geometry > Circles

The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. If the y-intercept of the graph of $y = P(x)$ is 2, what is b?

(A) -11 **(B)** -10 **(C)** -9 **(D)** 1 **(E)** 5

2001 AMC 12, Problem #19—
"The sum and product of the zeros of $P(x)$ are $-a$ and $-c$, respectively."

Solution
(A) The sum and product of the zeros of $P(x)$ are $-a$ and $-c$, respectively. Therefore,

$$-\frac{a}{3} = -c = 1 + a + b + c.$$

Since $c = P(0)$ is the y-intercept of $y = P(x)$, it follows that $c = 2$. Thus $a = 6$ and $b = -11$.

Difficulty: Hard
NCTM Standard: Algebra Standard for Grades 9–12: Use mathematical models to represent and understand quantitative relationships.
Mathworld.com Classification: Algebra > Polynomials

Points $A = (3,9)$, $B = (1,1)$, $C = (5,3)$, and $D = (a,b)$ lie in the first quadrant and are the vertices of quadrilateral $ABCD$. The quadrilateral formed by joining the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ is a square. What is the sum of the coordinates of point D?

(A) 7 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 16

2001 AMC 12, Problem #20—
"Sketch the figure and explore."

Solution

(C) Let the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ be denoted M,N,P,and Q, respectively. Then $M = (2,5)$ and $N = (3,2)$. Since $\overline{MN}$ has slope -3, the slope of $\overline{MQ}$ must be 1/3, and $MQ = MN = \sqrt{10}$. An equation for the line containing $\overline{MQ}$ is thus $y - 5 = \frac{1}{3}(x-2)$, or $y = (x+13)/3$. So Q has coordinates of the form $(a, \frac{1}{3}(a+13))$. Since $MQ = \sqrt{10}$, we have

$$\begin{aligned}(a-2)^2 + (\frac{a+13}{3} - 5)^2 &= 10\\ (a-2)^2 + (\frac{a-2}{3})^2 &= 10\\ \frac{10}{9}(a-2)^2 &= 10\\ (a-2)^2 &= 9\\ a-2 &= \pm 3\end{aligned}$$

Since Q is in the first quadrant, $a = 5$ and $Q = (5,6)$. Since Q is the midpoint of $\overline{AD}$ and $A = (3,9)$, we have $D = (7,3)$, and $7+3 = 10$.

OR

Use translation vectors. As before, $M = (2,5)$ and $N = (3,2)$. So $\overrightarrow{NM} = \langle -1,3 \rangle$. The vector $\overrightarrow{MQ}$ must have the same length as $\overrightarrow{MN}$ and be perpendicular to it, so $\overrightarrow{MQ} = \langle 3,1 \rangle$. Thus, $Q = (5,6)$. As before, $D = (7,3)$, and the answer is 10.

OR

Each pair of opposite sides of the square are parallel to a diagonal of $ABCD$, so the diagonals of $ABCD$ are perpendicular. Similarly, each pair of opposite sides of the square has length half that of a diagonal, so the diagonals of $ABCD$ are congruent. Since the slope of $\overline{AC}$ is -3 and $\overline{AC}$ is perpendicular to $\overline{BD}$, we have

$$\frac{b-1}{a-1} = \frac{1}{3}, \text{ so } a-1 = 3(b-1).$$

Since $AC = BD$,

$$40 = (a-1)^2 + (b-1)^2 = 9(b-1)^2 + (b-1)^2 = 10(b-1)^2,$$

and since b is positive, $b = 3$ and $a = 1 + 3(b-1) = 7$. So the answer is 10.

Difficulty: Hard
NCTM Standard: Geometry Standard for Grades 9–12: Use Cartesian coordinates and other coordinate systems, such as navigational, polar, or spherical systems, to analyze geometric situations.
Mathworld.com Classification: Geometry > Plane Geometry > Quadrilaterals
Geometry > Plane Geometry > Squares

Four positive integers a, b,c, and d have a product of 8! and satisfy

$$\begin{aligned} ab + a + b &= 524, \\ bc + b + c &= 146, and \\ cd + c + d &= 104. \end{aligned}$$

What is $a - d$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

2001 AMC 12, Problem #21— "$ab + a + b = (a+1)(b+1) - 1$."

Solution

(D) Note that

$$(a+1)(b+1) = ab + a + b + 1 = 524 + 1 = 525 = 3 \cdot 5^2 \cdot 7,$$

and

$$(b+1)(c+1) = bc + b + c + 1 = 146 + 1 = 147 = 3 \cdot 7^2.$$

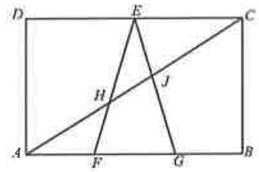
Since $(a+1)(b+1)$ is a multiple of 25 and $(b+1)(c+1)$ is not a multiple of 5, it follows that $a+1$ must be a multiple of 25. Since $a+1$ divides 525, a is one of 24, 74, 174, or 524. Among these only 24 is a divisor of 8!, so $a = 24$. This implies that $b + 1 = 21$, and $b = 20$. From this it follows that $c + 1 = 7$ and $c = 6$. Finally, $(c+1)(d+1) = 105 = 3 \cdot 5 \cdot 7$, so $d + 1 = 15$ and $d = 14$. Therefore, $a - d = 24 - 14 = 10$.

Difficulty: Hard

NCTM Standard: Algebra Standard for Grades 9–12: Represent and analyze mathematical situations and structures using algebraic symbols.

Mathworld.com Classification: Calculus and Analysis > Special Functions > Factorials

In rectangle $ABCD$, points F and G lie on $\overline{AB}$ so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of rectangle $ABCD$ is 70. Find the area of triangle EHJ.



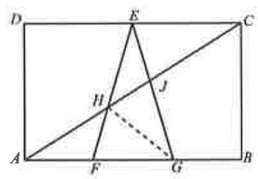
(A) $\frac{5}{2}$ **(B)** $\frac{35}{12}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** $\frac{35}{8}$

2001 AMC 12, Problem #22—

"$\triangle AFH$ and $\triangle CEH$ are similar. $\triangle AGJ$ and $\triangle CEJ$ are similar."

Solution

(C) The area of triangle EFG is $(1/6)(70) = 35/3$. Triangles AFH and CEH are similar, so $3/2 = EC/AF = EH/HF$ and $EH/EF = 3/5$. Triangles AGJ and CEJ are similar, so $3/4 = EC/AG = EJ/JG$ and $EJ/EG = 3/7$.



Since the areas of the triangles that have a common altitude are proportional to their bases, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EHG$ is 3/7, and the ratio of the area of $\triangle EHG$ to that of $\triangle EFG$ is 3/5. Therefore, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EFG$ is $(3/5)(3/7) = 9/35$. Thus, the area of $\triangle EHJ$ is $(9/35)(35/3) = 3$.

Difficulty: Hard

NCTM Standard: Geometry Standard for Grades 9–12: Explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them.

Mathworld.com Classification: Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles > Similar Triangle

A polynomial of degree four with leading coefficient 1 and integer coefficients has two real zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

(A) $\frac{1+i\sqrt{11}}{2}$ **(B)** $\frac{1+i}{2}$ **(C)** $\frac{1}{2}+i$ **(D)** $1+\frac{i}{2}$ **(E)** $\frac{1+i\sqrt{13}}{2}$

2001 AMC 12, Problem #23—
"Imaginary root always come in pairs."

Solution

(A) If r and s are the integer zeros, the polynomial can be written in the form

$$P(x) = (x-r)(x-s)(x^2+\alpha x+\beta).$$

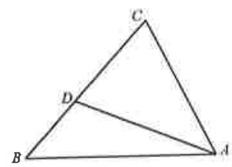
The coefficient of x^3, $\alpha-(r+s)$, is an integer, so α is an integer. The coefficient of x^2, $\beta-\alpha(r+s)+rs$, is an integer, so β is also an integer. Applying the quadratic formula gives the remaining zeros as

$$\frac{1}{2}(-\alpha \pm \sqrt{\alpha^2-4\beta}) = -\frac{\alpha}{2} \pm i\frac{\sqrt{4\beta-\alpha^2}}{2}.$$

Answer choices (A), (B), (C), and (E) require that $\alpha = -1$, which implies that the imaginary parts of the remaining zeros have the form $\pm\sqrt{4\beta-1}/2$. This is true only for choice (A).
Note that choice (D) is not possible since this choice requires $\alpha=-2$, which produces an imaginary part of the form $\sqrt{\beta-1}$, which cannot be $\frac{1}{2}$.

Difficulty: Hard
NCTM Standard: Algebra Standard for Grades 9–12: Represent and analyze mathematical situations and structures using algebraic symbols.
Mathworld.com Classification: Calculus and Analysis > Roots
Algebra > Polynomials

In triangle ABC, $\angle ABC = 45°$. Point D is on $\overline{BC}$ so that $2 \cdot BD = CD$ and $\angle DAB = 15°$. Find $\angle ACB$.



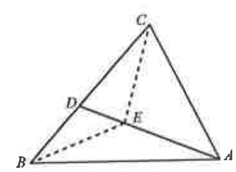
(A) $54°$ **(B)** $60°$ **(C)** $72°$ **(D)** $75°$ **(E)** $90°$

2001 AMC 12, Problem #24—

"Let E be a point on $\overline{AD}$ such that $\overline{CE}$ is perpendicular to $\overline{AD}$, and connect $\overline{BE}$. Explore the new figure."

Solution

(D) Let E be a point on $\overline{AD}$ such that $\overline{CE}$ is perpendicular to $\overline{AD}$, and draw $\overline{BE}$. Since $\angle ADC$ is an exterior angle of $\triangle ADB$ it follows that



$$\angle ADC = \angle DAB + \angle ABD = 15° + 45° = 60°.$$

Thus, $\triangle CDE$ is a $30° - 60° - 90°$ triangle and $DE = \frac{1}{2}CD = BD$. Hence, $\triangle BDE$ is isosceles and $\angle EBD = \angle BED = 30°$. But $\angle ECB$ is also equal to $30°$ and therefore $\triangle BEC$ is isosceles with $BE = EC$. On the other hand,

$$\angle ABE = \angle ABD - \angle EBD = 45° - 30° = 15° = \angle EAB.$$

Thus, $\triangle ABE$ is isosceles with $AE = BE$. Hence $AE = BE = EC$. The right triangle AEC is also isosceles with $\angle EAC = \angle ECA = 45°$. Hence,

$$\angle ACB = \angle ECA + \angle ECD = 45° + 30° = 75°.$$

Difficulty: Hard
NCTM Standard: Geometry Standard for Grades 9–12: Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.
Mathworld.com Classification: Geometry > Trigonometry > Angles > Exterior Angle
Geometry > Plane Geometry > Triangles > Special Triangles > Other Triangles

Consider sequences of positive real numbers of the form x, 2000, y, …, in which every term after the first is 1 less than the product of its two immediate neighbors. For how many different values of x does the term 2001 appear somewhere in the sequence?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** more than 4

2001 AMC 12, Problem #25—

"If a, b, and c are three consecutive terms of such a sequence, then $ac - 1 = b$, which can be rewritten as $c = (1+b)/a$."

Solution

(D) If a, b, and c are three consecutive terms of such a sequence, then $ac - 1 = b$, which can be rewritten as $c = (1+b)/a$. Applying this rule recursively and simplifying yields

$$\ldots, a, b, \frac{1+b}{a}, \frac{1+a+b}{ab}, \frac{1+a}{b}, a, b, \ldots$$

This shows that at most five different terms can appear in such a sequence. Moreover, the value of a is determined once the value 2000 is assigned to b and the value 2001 is assigned to another of the first five terms. Thus, there are four such sequences that contain 2001 s a term, namely

$$2001, 2000, 1, \frac{1}{1000}, \frac{1001}{1000}, 2001, \ldots,$$
$$1, 2000, 2001, \frac{1001}{1000}, \frac{1}{1000}, 1, \ldots,$$
$$\frac{2001}{4001999}, 2000, 4001999, 2001, \frac{2002}{4001999}, \frac{2001}{4001999}, \ldots, \text{ and}$$
$$4001999, 2000, \frac{2001}{4001999}, \frac{2002}{4001999}, 2001, 4001999, \ldots,$$

respectively. The four values of x are 2001, 1, $\frac{2001}{4001999}$, and 4001999.

Difficulty: Hard

NCTM Standard: Algebra Standard for Grades 9–12: Understand patterns, relations, and functions.

Mathworld.com Classification: Number Theory > Sequences